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SOME BOUNDARY-VALUE PROBLEMS OF THE ELASTIC AND THERMOELASTIC EQUILIBRIUM OF WEDGE-SHAPED BODIES†

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An exact solution is constructed of some boundary-value problems of the thermoelastic and elastic equilibrium of wedge-shaped bodies, bounded by two infinite or finite coordinate planes, that is, by the faces of a dihedral angle, with rotationally-symmetric orthogonal coordinates. In the case when the wedge is infinite, a steady temperature field and corresponding surface perturbations act on it. If the wedge-shaped body occupies a finite domain, bounded by the coordinate surfaces of one of the rotationally-symmetric systems of coordinates, then surface perturbations are specified on its faces (when there is no temperature field) and homogeneous conditions of a special form are satisfied on the remaining part of the surface. The surface perturbations on each of the two faces correspond to the specification: (a) displacements, (b) tangential displacements and a normal stress and (c) shear stresses and a normal displacement.

In the analysis of a boundary-value problem of the elastic equilibrium of an infinite wedge [1, 2], a solution was constructed in a cylindrical system of coordinates (when there was no temperature perturbation) using formulae for a double integral transformation and, in particular, using a Kontorovich–Lebedev transformation [1] along a radial coordinate and a Fourier transformation along one of the other linear coordinates. Either displacements, tangential displacements or shear stresses and a normal displacement were specified on the wedge faces.

1. FORMULATION OF THE PROBLEM

Using the method of separation of variables and double series, the solutions of static boundary-value problems are constructed in the theory of elasticity for a curvilinear coordinate parallelepiped

$$\Omega = \{\rho_0 < \rho < \rho_1, 0 < \alpha < \alpha_1, \beta_0 < \beta < \beta_1\} \tag{1.1}$$

where ρ, α, β are rotationally symmetric orthogonal coordinates with Lamé coefficients $h_\rho = h_\beta = h(\rho, \beta)$, $h_\alpha = H(\rho, \beta)$ (cylindrical coordinates, spherical coordinates, prolate spheroidal coordinates, oblate spheroidal coordinates, paraboloidal coordinates, toroidal coordinates and bispherical coordinates belong to fundamental rotationally symmetric systems of coordinates [3]). In the planes $\alpha = 0$ and $\alpha = \alpha_1$, one specifies (a) displacements, (b) tangential displacements and the normal stress and (c) shear stresses and the normal displacement. Homogeneous conditions of a special type are specified on the lateral surfaces ($\rho = \rho_0$, $\rho = \rho_1$, $\beta = \beta_0$, $\beta = \beta_1$). A way of extending the method of solution to find the thermoelastic equilibrium of an infinite wedge will be indicated below.

The problem of the elastic equilibrium of a wedge is generalized (the generalization consists of the possibility of constructing the solution in any of the rotationally symmetric systems of coordinates and not only in a circular cylindrical system of coordinates) although the methods for solving it are also simplified by: (1) constructing a general solution for the class of boundary-value problem in thermoelasticity being

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considered and (2) replacing the classical conditions, specified on the boundary surfaces α = const, by equivalent conditions.

The following can be said about the effectiveness of the solutions. If, by using the method of separation of variables, a solution can be effectively constructed for the principal boundary-value problems for Laplace's equation with zero conditions when $\rho = \rho_j$ and $\beta = \beta_j$, where $j = 0, 1$, then the elastic equilibrium of the bodies being considered here can be found with the same efficiency, in the same domain (1.1) and by the same method.

2. THE EQUILIBRIUM EQUATIONS, EQUATIONS OF STATE AND BOUNDARY CONDITIONS

If the temperature field is independent of time and there are no mass forces, the thermoelastic equilibrium of a homogeneous, isotropic body in rotationally symmetric orthogonal systems of coordinates can be represented in the form [4]

$$\Delta[2(\kappa-2)\mathbf{U}+R(2\operatorname{div}\mathbf{U}-(8-\kappa)kT)] = 0 \tag{2.1}$$

or in the form

$$\begin{gathered}\operatorname{grad}[\kappa\operatorname{div}\mathbf{U}-(8-\kappa)kT]-(\kappa-2)\operatorname{rot}\operatorname{rot}\mathbf{U} = 0, \quad \operatorname{div}\operatorname{rot}\mathbf{U} = 0\\ \mathbf{U} = u_1\mathbf{l}_1+u_2\mathbf{l}_2+u_3\mathbf{l}_3, \quad R = x\mathbf{l}_1+y\mathbf{l}_2+z\mathbf{l}_3, \quad \kappa = 4(1-\nu)\end{gathered} \tag{2.2}$$

where $\mathbf{U}$ is the displacement vector, and u_1, u_2 and u_3 are its components along the axes of the Cartesian system of coordinates x, y and z, ν is Poisson's ratio and k is the coefficient of linear thermal expansion.

The components of the stress tensor can be expressed in terms of the displacements as follows:

$$\frac{h^2}{\mu}R^{(\rho)} = \eta_1 h^2\operatorname{div}\mathbf{U}+2hu_\rho+2wh_\beta-\eta_2h^2T, \quad \frac{hH}{\mu}A^{(\rho)} = H^2\left(\frac{\upsilon}{H}\right)_\rho+hu_\alpha$$

$$\frac{hH}{\mu}A^{(\alpha)} = \eta_1hH\operatorname{div}\mathbf{U}+2h\upsilon_\alpha+2wH_\beta+2uH_\rho-\eta_2hHT, \quad \frac{hH}{\mu}A^{(\beta)} = hw_\alpha+H^2\left(\frac{\upsilon}{H}\right)_\beta$$

$$\frac{h^2}{\mu}B^{(\beta)} = h_1\operatorname{div}\mathbf{U}+2hw_\beta+2uh_\rho-\eta_2h^2T, \quad \frac{1}{\mu}R^{(\beta)} = \left(\frac{u}{h}\right)_\beta+\left(\frac{w}{h}\right)_\rho$$

$$\operatorname{div}\mathbf{U} = \frac{(hHu)_\rho+h^2\upsilon_\alpha+(hHw)_\beta}{h^2H}; \quad \mu = \frac{E}{2(1+\nu)}, \quad \eta_1 = \frac{4-\kappa}{\kappa-2}, \quad \eta_2 = k\frac{8-\kappa}{\kappa-2}$$

Partial derivatives with respect to the corresponding coordinates are denoted by the subscripts ρ, α, β, $R^{(\rho)}, A^{(\alpha)}, B^{(\beta)}$ are the normal stresses and $R^{(\alpha)} = A^{(\rho)}, R^{(\beta)} = B^{(\rho)}, A^{(\beta)} = B^{(\alpha)}$ are the shear stresses, u, υ and w are the components of the displacement vector $\mathbf{U}$ along the tangents to the coordinates lines ρ, α, β, E is the modulus of elasticity and T is the change in temperature in the elastic body, which obeys the equation

$$\Delta T = \frac{(HT_\rho)_\rho+(HT_\beta)_\beta}{h^2H}+\frac{T_{\alpha\alpha}}{H^2} = 0 \tag{2.3}$$

and the corresponding boundary conditions. In the case of spherical coordinates r, α, β, $h = r$, $H = r\sin\beta$ and the operation $\partial/\partial\rho$ is replaced by the operation $r\partial/\partial r$.

In rotationally symmetric coordinates, the equation

$$\Delta[(\kappa-2)H\operatorname{rot}^{(\alpha)}\mathbf{U}+z(\kappa\operatorname{div}\mathbf{U}-(8-\kappa)kT)] = 0 \tag{2.4}$$

can be obtained from system (2.2), where $\operatorname{rot}^{(\alpha)}\mathbf{U}$ is the projection of the vector $\operatorname{rot}\mathbf{U}$ onto the tangent to the coordinate line α.

Introducing the notation

$$L^{[\rho\kappa]}(hw) - L^{[\beta\kappa]}(hu) - (4-\kappa/2)kh^2 zT = \kappa h^2 K$$

$$h^{-2}[L^{[\rho(1-\kappa)]}(hu) + L^{[\beta(1-\kappa)]}(hw)] + \upsilon_\alpha - (4-\kappa/2)kHT = (\kappa-2)D \quad (2.5)$$

$$L^{[\gamma s]}(f) = H^{s+1}(H^{-s}f)_\gamma$$

and using relations (2.1), (2.2) and (2.4), we obtain the equations

$$\Delta K = 0, \quad \Delta(D\cos\alpha - \upsilon\sin\alpha) = 0, \quad \Delta(D\sin\alpha + \upsilon\cos\alpha) = 0$$

$$L^{[1\rho\beta]}(T) = \frac{\eta_2}{2}[\kappa L^{[\beta(\kappa-1)]}(HT) + (\kappa-2)L^{[\rho(-\kappa)]}(zT) - 2H^2 T_\beta]$$

$$L^{[2\rho\beta]}(T) = \frac{\eta_2}{2}[\kappa L^{[\rho(\kappa-1)]}(HT) - (\kappa-2)L^{[\beta(-\kappa)]}(zT) - 2H^2 T_\rho] \quad (2.6)$$

$$(hw)_{\alpha\alpha} + \kappa^2(hw) = L^{[\beta(\kappa-1)]}(\upsilon_\alpha - \kappa D) - \kappa L^{[\rho(-\kappa)]}(K) - L^{(1\rho\beta)}(T)$$

$$(hu)_{\alpha\alpha} + \kappa^2(hu) = L^{[\rho(\kappa-1)]}(\upsilon_\alpha - \kappa D) + \kappa L^{[\beta(-\kappa)]}(K) - L^{(2\rho\beta)}(T)$$

Note that the second and third equations in system (2.6) can be written in the form

$$\Delta D - H^{-2}D - 2H^{-2}\upsilon_\alpha = 0, \quad \Delta\upsilon - H^{-2}\upsilon + 2H^{-2}D_\alpha = 0$$

In the case of the twisting of solids of revolution, when $\upsilon = \upsilon(\rho, \beta)$, $w = 0$, $u = 0$, we have

$$\Delta_2\upsilon - H^2\upsilon = H^{-1}h^{-2}[(H\upsilon_\rho)_\rho + (H\upsilon_\beta)_\beta] - H^2\upsilon = 0$$

In the case of an axisymmetric stressed state, when

$$\upsilon = 0, \quad T = T(\alpha, \beta)\ (\Delta_2 T = 0), \quad w = w(\rho, \beta), \quad u = u(\rho, \beta)$$

the following form of solution

$$\Delta_2 K = 0, \quad \Delta_2 D - H^{-2}D = 0$$

$$hw = -\frac{1}{\kappa}[L^{[\beta(\kappa-1)]}(D) + L^{[\rho(-\kappa)]}(K)] - \frac{1}{\kappa^2}L^{[1\rho\beta]}(T)$$

$$hu = -\frac{1}{\kappa}[L^{[\rho(\kappa-1)]}(D) - L^{[\beta(-\kappa)]}(K)] - \frac{1}{\kappa^2}L^{[2\rho\beta]}(T)$$

is obtained from Eqs (2.6).

It should be noted that problems of the elastic (and not thermoelastic) equilibrium of a curvilinear coordinate parallelepiped (CCP) occupying the domain (1.1) will be solved next, although the procedure for finding the solution of boundary-value problems in the theory of elasticity for a CCP can be directly transferred to finding the thermoelastic equilibrium of an infinite wedge. It can therefore be assumed that, if it is possible to substantiate and use the corresponding integral transforms (in the case of cylindrical coordinates r, α, z, these will be a Kontorovich–Lebedev transformation with respect to the coordinate r and a Fourier transformation with respect to the coordinate z), then the solutions of the corresponding boundary-value problems of thermoelasticity can also be efficiently constructed in the case of an infinite wedge. In the case of circular cylindrical coordinates r, α, z, what has been said also holds for the domain

$$\Omega = \{0 < r < \infty, 0 < \alpha < \alpha_1, 0 < z < z_1\}$$

which is only infinite along the r coordinate, and then the symmetry conditions $T_\alpha = 0, \upsilon = 0, A^{(z)} = 0$, $A^{(r)} = 0$ or antisymmetry conditions $I = 0, A^{(\alpha)} = 0, w = 0, u = 0$ are satisfied when $z = z_j$ $(j = 0, 1)$ (in this case, along the z coordinate instead of a Fourier transformation, we will have a corresponding trigonometric Fourier series).

The elastic (and not thermoelastic) equilibrium of a CCP will next be considered. The boundary conditions in this case have the following form

$$\begin{aligned} \text{when } \rho = \rho_j\text{: } & a)\ L^{[\rho(1-\kappa)]}(hu) = 0, \quad \upsilon = 0, \quad w = 0 \text{ or} \\ & b)\ u = 0, \quad L^{[\rho(\kappa-1)]}(\upsilon) = 0, \quad L^{[\rho\kappa]}(hw) = 0 \end{aligned} \tag{2.7}$$

$$\begin{aligned} \text{when } \beta = \beta_j\text{: } & a)\ L^{[\beta(1-\kappa)]}(hw) = 0, \quad u = 0, \quad \upsilon = 0 \text{ or} \\ & b)\ w = 0, \quad L^{[\beta\kappa]}(hu) = 0, \quad L^{[\beta(\kappa-1)]}(\upsilon) = 0 \end{aligned} \tag{2.8}$$

$$\begin{aligned} \text{when } \alpha = \alpha_j\text{: } & a)\ \upsilon = f_{j1}(\rho, \beta), \quad A^{(\beta)} = F_{j2}(\rho, \beta), \quad A^{(\rho)} = F_{j3}(\rho, \beta) \text{ or} \\ & b)\ A^{(\alpha)} = F_{j1}(\rho, \beta), \quad hw = f_{j2}(\rho, \beta), \quad hu = f_{j3}(\rho, \beta) \text{ or} \\ & c)\ \upsilon = f_{j1}(\rho, \beta), \quad hw = f_{j2}(\rho, \beta), \quad hu = f_{j3}(\rho, \beta) \end{aligned} \tag{2.9}$$

In formulae (2.7)–(2.9), $j = 0, 1$ and $\alpha_0 = 0$. The condition imposed on the functions f_{jl} and F_{jl} $(l = 1, 2, 3)$ will be discussed below and we will only mention now that these functions are such that compatibility conditions are satisfied on the edges of the CCP.

Note that the smaller the curvature of the boundary surface $\rho = \rho_j$, the lesser the extent to which conditions (2.7) for cases a and b respectively differ from the conditions

$$\begin{aligned} \text{when } \rho = \rho_j\text{: } & a)\ R^{(\rho)} = 0, \quad \upsilon = 0, \quad w = 0 \text{ and} \\ & b)\ u = 0, \quad A^{[\rho]} = 0, \quad R^{[\beta]} = 0 \end{aligned} \tag{2.10}$$

Conditions (2.7)a and (2.7)b are equivalent to the corresponding conditions (2.10) when $\rho = \rho_j$ is a plane. What has been said above also holds for the surface $\beta = \beta_j$ and conditions (2.8).

3. THE GENERAL SOLUTION AND TRANSFORMATION OF THE BOUNDARY CONDITIONS

From the first three equations of system (2.6), it follows that

$$K = \varphi_1, \quad D = \varphi_2 \cos\alpha + \varphi_3 \sin\alpha, \quad \upsilon = \varphi_3 \cos\alpha - \varphi_2 \sin\alpha \tag{3.1}$$

where φ_1, φ_2 and φ_3 are harmonic functions, and w and u are expressed in terms of φ_1, φ_2 and φ_3 by integrating the last two equations of (2.6).

Using the method of separation of variables and taking account of conditions (2.7)–(2.9), we represent the harmonic functions φ_1, φ_2 and φ_3 in the following form (henceforth, summation is carried out from $n = 1$ to $n = \infty$ and from $m = 1$ to $m = \infty$ everywhere)

$$\begin{aligned} & \varphi_l = \sum_{n, m} \Phi_{lmn}(\alpha) \Psi_{lmn}(\rho, \beta) \\ & \Phi_{lmn} = A_{lmn} e^{-p_l \alpha} + B_{lmn} e^{p_l(\alpha - \alpha_1)}, \quad l = 1, 2, 3, \quad p_l = p_l(m, n), \quad p_2 = p_3 \end{aligned} \tag{3.2}$$

A_{lmn} and B_{lmn} are constants, and Ψ_{lmn} is a non-trivial solution of the following Sturm–Liouville problem [5]

$$[H(\Psi_{lmn})_\rho]_\rho + [H(\Psi_{lmn})_\beta]_\beta + \frac{(hp_l)^2}{H} \Psi_{lmn} = 0 \tag{3.3}$$

$$\text{when} \quad \rho = \rho_j\text{: } a)\ \Psi_{lmn} = 0 \quad \text{or} \quad b)\ L^{[\rho\kappa_l]}(\Psi_{lmn}) = 0 \tag{3.4}$$

$$\text{when} \quad \beta = \beta_j\text{: } a)\ \Psi_{lmn} = 0 \quad \text{or} \quad b)\ L^{[\beta\kappa_l]}(\Psi_{lmn}) = 0 \tag{3.5}$$

Here, $\kappa_1 = -\kappa$, $\kappa_2 = \kappa_3 = \kappa - 1$; $\Psi_{2mn} = \Psi_{3mn}$ since $\kappa_2 = \kappa_3$. We also note here that, when $\rho = \rho_j$, the following holds: if $\Psi_{1mn} = 0$, then $L^{[\rho(\kappa-1)]}(\Psi_{2mn}) = 0$, $L^{[\rho(\kappa-1)]}(\Psi_{3mn}) = 0$. If, however, $L^{[\rho(-\kappa)]}(\Psi_{1mn}) = 0$, then $\Psi_{2mn} = \Psi_{3mn} = 0$. Similar relations hold when $\beta = \beta_j$ (when ρ is replaced by β).

It is understood that boundary-value problem (3.3)–(3.5) in considered in cylindrical, spherical paraboloidal, toroidal and bispherical coordinates as well as in prolate and oblate spheroidal coordinates. In all of these coordinates, subsequent separation of the variables in Eq. (3.3) is possible, which leads to well-known (and studied) regular one-dimensional Sturm–Liouville problems. Depending on the system of coordinates, the functions $\Psi_{lmn}(\rho, \beta)$ are a product: (1) of Bessel functions and trigonometrical functions, (2) Legendre functions and trigonometrical functions, (3) Bessel functions and Bessel functions, (4) Legendre functions and Legendre functions (more precisely, we mean by Bessel functions and Legendre functions linear combinations of Bessel functions and linear combinations of Legendre functions, respectively) [6]. In particular, if a CCP is treated in a cylindrical system of coordinates r, α, z and conditions (2.7) and (2.8) are satisfied on the lateral surfaces of the CCP in case a, then

$$\Psi_{2mn} = \Psi_{3mn} = [I_{i\bar{n}}^{(R)}(r_m)K_{i\bar{n}}(r_{0m}) - K_{i\bar{n}}(r_m)I_{i\bar{n}}^{(R)}(r_{0m})]\sin z_m = R_0(r)\sin z_m$$

$$\Psi_{1mn} = \{I_{i\tilde{n}}^{(R)}(r_m)[L^{[r(-\kappa)]}(K_{i\tilde{n}}(r_m))]_{r=r_0} -$$
$$- K_{i\tilde{n}}(r_m)[L^{[r(-\kappa)]}(I_{i\tilde{n}}^{(R)}(r_m))]_{r=r_0}\}\cos z_m = R_1(r)\cos z_m$$

Here

$$r_m = \frac{\pi m}{z_1}r, \quad r_{0m} = \frac{\pi m}{z_1}r_0, \quad z_m = \frac{\pi m}{z_1}z$$

$K_{ia}(r_m)$ is a McDonald function with an imaginary index and $a = \bar{n}$ or $a = \tilde{n}$, $I_{ia}^{(R)}(r_m)$ is the real part of a modified Bessel function of the first kind with imaginary index and $\bar{n} = \bar{n}(m, n)$ and $\tilde{n} = \tilde{n}(m, n)$ are respectively the roots of number n of the transcendental equations

$$R_0(r_1) = 0 \text{ and } [L^{[r(-\kappa)]}(R_1(r))]_{r=r_1} = 0$$

Using relations (3.1), (3.2) and (2.6), for the CCP we will have

$$\upsilon = \sum_{n,m}(\Phi_{3mn}\cos\alpha - \Phi_{2mn}\sin\alpha)\Psi_{2mn}$$

$$\begin{aligned} hw &= \sum_{n,m}\left[\Phi_{mn}L^{[\beta(\kappa-1)]}(\Psi_{2mn}) - \frac{\kappa}{p_1^2+\kappa^2}\Phi_{1mn}L^{[\rho(-\kappa)]}(\Psi_{1mn})\right] + hw_0 \\ hu &= \sum_{n,m}\left[\Phi_{mn}L^{[\rho(\kappa-1)]}(\Psi_{2mn}) + \frac{\kappa}{p_1^2+\kappa^2}\Phi_{1mn}L^{[\beta(-\kappa)]}(\Psi_{1mn})\right] + hu_0 \end{aligned} \tag{3.6}$$

$$\Phi_{mn} = \frac{\kappa-1}{p_2^2+(\kappa-1)^2}\left(\frac{\cos\alpha}{\kappa-1}\frac{d\Phi_{3mn}}{d\alpha} - \sin\alpha\Phi_{3mn} - \frac{\sin\alpha}{\kappa-1}\frac{d\Phi_{2mn}}{d\alpha} - \cos\alpha\Phi_{2mn}\right)$$

In relations (3.6), expressions for hw and hu are obtained by integrating the last two equations of (2.6), and hw_0 and hu_0 are the general solution of the corresponding homogeneous equations, which has the form

$$hw_0 = \psi_{31}(\rho, \beta)\cos\kappa\alpha + \psi_{32}(\rho, \beta)\sin\kappa\alpha, \quad hu_0 = \psi_{11}(\rho, \beta)\cos\kappa\alpha + \psi_{12}(\rho, \beta)\sin\kappa\alpha \tag{3.7}$$

If expression (3.6), taking formulae (3.7) into account, is substituted into the left-hand sides of equalities (2.5) (we recall that $T = 0$) and, taking formulae (3.2) account, equalities (3.1) are substituted into the right-hand sides of equalities (2.5), we obtain the following system of equations for the functions $\psi_{3s}(\rho, \beta)$ and $\psi_{1s}(\rho, \beta)$ $(s = 1, 2)$

$$L^{[\rho\kappa]}(\psi_{3s}) - L^{[\beta\kappa]}(\psi_{1s}) = 0, \quad L^{[\rho(1-\kappa)]}(\psi_{1s}) + L^{[\beta(1-\kappa)]}(\psi_{3s}) = 0 \tag{3.8}$$

For the class of boundary-value problems being considered

$$\psi_{3s} = 0, \quad \psi_{1s} = 0 \tag{3.9}$$

apart from in the two following cases:

(1) when $\rho = \rho_0$ and $\rho = \rho_1$, conditions (2.7)*a* are specified, and when $\beta = \beta_0$ and $\beta = \beta_1$ conditions (2.8)*b* are specified. The Lamé coefficient H can be represented in the form $H = H_1(\rho)H_2(\beta)$. In this case

$$\psi_{11} = A_1 H_1^{1-\kappa} H_2^{\kappa}, \quad \psi_{12} = B_1 H_1^{1-\kappa} H_2^{\kappa}, \quad \psi_{31} = \psi_{32} = 0$$

and thereby

$$hu = [A_1\cos(\kappa\alpha) + B_1\sin(\kappa\alpha)]H_1^{1-\kappa}H_2^{\kappa}, \quad \upsilon = 0, \quad w = 0 \tag{3.10}$$

where A_1 and B_1 are constants;

(2) when $\rho = \rho_0$ and $\rho = \rho_1$, conditions (2.7)*b* are specified, and when $\beta = \beta_0$ and $\beta = \beta_1$ conditions (2.8)*b* are specified and $H = H_1(\rho)H_2(\beta)$. In this case,

$$\psi_{11} = 0, \quad \psi_{12} = 0, \quad \psi_{31} = A_3 H_1^{\kappa} H_2^{1-\kappa}, \quad \psi_{32} = B_3 H_1^{\kappa} H_2^{1-\kappa}$$

and by the same token

$$u = 0, \quad \upsilon = 0, \quad hw = [A_3\cos(\kappa\alpha) + B_3\sin(\kappa\alpha)]H_1^{\kappa}H_2^{1-\kappa} \tag{3.11}$$

where A_3 and B_3 are constants.

For all the remaining cases, the solution of the system

$$(H^{-\kappa}\psi_{3s})_\rho - (H^{-\kappa}\psi_{1s})_\beta = 0, \quad (H^{\kappa-1}\psi_{1s})_\rho + (H^{\kappa-1}\psi_{3s})_\beta = 0 \tag{3.12}$$

obtained from system (3.8) with boundary conditions which are determined by the values of hw and hu on the edges $\alpha = 0$ and $\alpha = \alpha_1$, will be $\psi_{3s} = 0$, $\psi_{1s} = 0$.

In fact, by taking

$$\psi_{3s} = -H^{-\kappa+1}(H^{\kappa-1/2}\Psi)_\rho, \quad \psi_{1s} = H^{-\kappa+1}(H^{\kappa-1/2}\Psi)_\beta \tag{3.13}$$

we arrive at the following boundary-value problem

$$\Psi_{\rho\rho} + \Psi_{\beta\beta} - (\kappa^2 - 1/4)h^2H^{-2}\Psi = 0 \tag{3.14}$$

when $\rho = \rho_j$: $(H^{\kappa-1/2}\Psi)_\rho = 0$ or $(H^{\kappa-1/2}\Psi)_\beta = 0$
when $\beta = \beta_j$: $(H^{\kappa-1/2}\Psi)_\beta = 0$ or $(H^{\kappa-1/2}\Psi)_\rho = 0$.

This problem always has only the solution $\Psi = c_0H^{1/2-\kappa}$ but substitution of this value of Ψ into relations (3.13) gives $\psi_{3s} = 0$, $\psi_{1s} = 0$ or, what is the same thing, $hw = 0$, $hu = 0$. The constant c_0 can therefore always be assumed to be zero.

We now represent conditions (2.9) in the form
when $\alpha = \alpha_j$:

$$a)\ \upsilon = f_{j1}(\rho, \beta), \quad L^{[\rho(1-\kappa)]}[(hu)_\alpha] + L^{[\beta(1-\kappa)]}[(hw)_\alpha] = h^2\tilde{F}_{j2}(\rho, \beta)$$

$$L^{[\rho\kappa]}[(hw)_\alpha] - L^{[\beta\kappa]}[(hu)_\alpha] = h^2\tilde{F}_{j3}(\rho, \beta) \quad \text{or}$$

$$b)\ v_\alpha = \tilde{F}_{j1}(\rho,\beta),\quad L^{[\rho(1-\kappa)]}(hu) + L^{[\beta(1-\kappa)]}(hw) = h^2\tilde{f}_{j2}(\rho,\beta)$$
$$L^{[\rho\kappa]}(hw) - L^{[\beta\kappa]}(hu) = h^2\tilde{f}_{j3}(\rho,\beta) \text{ or} \tag{3.15}$$
$$c)\ v = f_{j1}(\rho,\beta),\quad L^{[\rho(1-\kappa)]}(hu) + L^{[\beta(1-\kappa)]}(hw) = h^2\tilde{f}_{j2}(\rho,\beta)$$
$$L^{[\rho\kappa]}(hw) - L^{[\beta\kappa]}(hu) = h^2\tilde{f}_{j3}(\rho,\beta)$$

The functions $\tilde{F}$ themselves and the functions f_{jl} ($l = 1, 2, 3$) together with their first derivatives are expanded in uniformly converging series in the eigenfunctions of problems (3.3)–(3.5) (the functions with a tilde are obtained as a result of corresponding operations on the functions F_{j2}, F_{j3}, f_{j2} and f_{j3}). In addition, certain additional requirements are imposed on the functions f_{jl} from conditions (2.9). For example, if boundary condition (2.6), (2.7)*a*, (2.8)*a*, (3.15)*b* is considered, then, apart from the fact that

$$f_{j2}(\rho_j,\beta) = 0,\quad [L^{[\beta(1-\kappa)]}(f_{j2})]_{\beta=\beta_j} = 0,\quad f_{j3}(\rho,\beta_j) = 0,\quad [L^{[\rho(1-\kappa)]}(f_{j3})]_{\rho=\rho_j} = 0$$

the requirements that $f_{j2}(\rho,\beta_j) = 0$ and $f_{j3}(\rho_j,\beta) = 0$ are additionally imposed on the functions f_{j2} and f_{j3}.

Proof of the equivalence of the boundary conditions (3.5) and (2.9) reduces to proof of the fact that system (3.12), with the corresponding boundary conditions, only has a trivial solution (with the exception of cases 1 and 2, which have been pointed out above) and the latter problem, in turn, reduces to an investigation of boundary-value problem (3.14). Finally, it can be stated that, if $H = H_1(\rho)H_2(\beta)$, the solution (3.10) has to be added to problem (2.6), (2.7)*a*, (2.8)*b*, (3.15), and the solution (3.11) to the problem (2.6), (2.7)*b*, (2.8)*a*, (3.15). In all of the remaining cases, conditions (3.15) and (2.9) are equivalent.

The aim of this paper is to construct a regular solution of boundary-value problems (2.6)–(2.8), (3.15) and we shall therefore define the concept of regularity.

We shall say that a solution of system (2.6), which is determined by the functions u, v and w, is regular if the functions u, v and w are triply continuously differentiable in the domain $\tilde{\Omega}$, where $\tilde{\Omega}$ is the domain Ω together with the boundaries $\rho = \rho_j$ and $\beta = \beta_j$, and, in the surface $\alpha = \alpha_j$, they can be represented, together with their first and second derivatives, by absolutely and uniformly converging Fourier series in the eigenfunctions of problem (3.3)–(3.5). In addition, we assume that the equilibrium equations hold when $\rho = \rho_j$ and $\beta = \beta_j$.

The advisability of replacing conditions (2.9) by conditions (3.15) is confirmed by the formulae

$$v = \varphi_3\cos\alpha - \varphi_2\sin\alpha,\quad v_\alpha = [(\varphi_3)_\alpha - \varphi_2]\cos\alpha - [\varphi_3 + (\varphi_2)_\alpha]\sin\alpha$$
$$h^{-2}[L^{[\rho(1-\kappa)]}(hu) + L^{[\beta(1-\kappa)]}(hw)] = [(\varphi_2)_\alpha + (\kappa-1)\varphi_3]\sin\alpha +$$
$$+[(\kappa-1)\varphi_2 - (\varphi_3)_\alpha]\cos\alpha$$
$$h^{-2}\{L^{[\rho(1-\kappa)]}[(hu)_\alpha] + L^{[\beta(1-\kappa)]}[(hw)_\alpha]\} = [(\varphi_2)_{\alpha\alpha} + (\kappa-1)\varphi_2 + \kappa(\varphi_3)_\alpha]\sin\alpha + \tag{3.16}$$
$$+[\kappa(\varphi_2)_\alpha - (\varphi_3)_{\alpha\alpha} + (\kappa-1)\varphi_3]\cos\alpha$$
$$h^{-2}[L^{[\rho\kappa]}(hw) - L^{[\beta\kappa]}(hu)] = \kappa\varphi_1,\quad h^{-2}\{L^{[\rho\kappa]}[(hw)_\alpha] - L^{[\beta\kappa]}[(hu)_\alpha]\} = \kappa(\varphi_1)_\alpha$$

in the right-hand sides of which the variables ρ and β do not appear in explicit form and there are no derivatives of these variables.

We note that, as previously [7], the overall elastic field, corresponding to the boundary-value problems being considered can be represented in the form of the sum of an elastic field with $D = 0$ and $v = 0$ or, what is the same thing, with $\varphi_2 = 0$ and $\varphi_3 = 0$ and an elastic field with $K = 0$, that is, with $\varphi_1 = 0$. The boundary conditions for the function $K = \varphi_1$ when $\alpha = 0$ and $\alpha = \alpha_1$ are determined by the three equalities in (3.15).

4. ANALYTICAL SOLUTION OF SOME BOUNDARY-VALUE PROBLEMS

By replacing conditions (2.9) by conditions (3.15) and using the representations (3.6) we can find a regular solution of any of the boundary-value problems (2.6)–(2.8), (3.15). We shall demonstrate this, taking the example of the regular solution of boundary-value problem (2.6), (2.7)*a*, (2.8)*a*, (2.9)*a* (we call this Problem G_0), by introducing the following boundary conditions

$$\begin{aligned} &a)\ 2v = f_{11}-f_{01},\quad 2A^{(\beta)} = F_{12}-F_{02},\quad 2A^{(\rho)} = F_{13}-F_{03}\quad \text{when}\quad \alpha = \alpha_1 \\ &b)\ 2v = -(f_{11}-f_{01}),\quad 2A^{(\beta)} = -(F_{12}-F_{02}),\quad 2A^{(\rho)} = -(F_{13}-F_{03})\quad \text{when}\quad \alpha = 0 \end{aligned} \tag{4.1}$$

$$2v = f_{11}+f_{01},\quad 2A^{(\beta)} = F_{12}+F_{02},\quad 2A^{(\rho)} = F_{13}+F_{03}\quad \text{when}\quad a)\ \alpha = \alpha_1,\ b)\ \alpha = 0 \tag{4.2}$$

We will represent Problem G_0 in the form of the sum of boundary-value problems (2.6), (2.7)*a*, (2.8)*a*, (4.1) (Problem G_1), and (2.6), (2.7)*a*, (2.8)*a*, (4.2) (Problem G_2). If the boundary conditions (2.9) and (3.15) with zero right-hand sides are denoted by $(2.9)^0$ and $(3.15)^0$, then the solutions of Problems G_1 and G_2 reduce respectively to the solution of problems (2.6), (2.7)*a*, (2.8)*a*, $(2.9)^0a$ when $\alpha = \alpha_1$, (4.1)*b*, and (2.6), (2.7)*a*, (2.8)*a*, $(2.9)^0b$ when $\alpha = \alpha_1$, (4.2)*b* in which, $\alpha_1/2$ is chosen instead of α_1. The method for solving these two problems is the same, and we shall therefore confine ourselves to solving the first of them, which is equivalent to problem (2.6), (2.7)*a*, (2.8)*a*, $(3.15)^0a$ when $\alpha = 0$, (3.15)*a*, by representing the functions φ_1, φ_2 and φ_3 as follows:

$$\varphi_l = \sum_{n,m} \Phi_{lmn}(\alpha)\Psi_{lmn}(\rho,\beta),\quad l = 1,2,3 \tag{4.3}$$

Here

$$\Phi_{1mn}(\alpha) = A_{1mn}\frac{\operatorname{ch}\tilde{p}\alpha}{\operatorname{ch}\tilde{p}\alpha_1},\quad \Phi_{2mn}(\alpha) = A_{2mn}\frac{\operatorname{ch}p\alpha}{\operatorname{ch}p\alpha_1},\quad \Phi_{3mn}(\alpha) = A_{3mn}\frac{\operatorname{sh}p\alpha}{\operatorname{ch}p\alpha_1}$$

$$p_2(m,n) = p_3(m,n) = p(m,n) = p,\quad p_1(m,n) = \tilde{p}(m,n) = \tilde{p}$$

$\Psi_{2mn}(\rho,\beta) = \Psi_{3mn}(\rho,\beta) = \Psi_{mn}(\rho,\beta)$ are the eigenfunctions of problem (3.3), (3.4)*a*, (3.5)*a*, and $\Psi_{1mn}(\rho,\beta)$ are are the eigenfunctions of problem (3.3), (3.4)*b*, (3.5)*b*.

Taking relations (4.43) and (3.16) into account, the constants A_{lmn} are determined from the following system of equations

$$\begin{aligned} &\kappa\operatorname{th}\tilde{p}\alpha_1 A_{1mn} = \tilde{F}_{13mn} \\ &[(p^2-\kappa+1)\operatorname{tg}\alpha_1+\operatorname{th}p\alpha_1]A_{2mn}-[(p^2-\kappa+1)\operatorname{th}p\alpha_1-\operatorname{tg}\alpha_1]A_{3mn} = \tilde{F}_{12mn}/\cos\alpha_1 \\ &\operatorname{tg}\alpha_1 A_{2mn}-\operatorname{th}p\alpha_1 A_{3mn} = -f_{11mn}/\cos\alpha_1 \end{aligned} \tag{4.4}$$

where f_{11mn} and $\tilde{F}_{12mn}$ are the Fourier coefficients of the functions $f_{11}(\rho,\beta)$ and $\tilde{F}_{12}(\rho,\beta)$, which have been expanded in series in the functions Ψ_{mn}, and $\tilde{F}_{13mn}$ are the Fourier coefficients of the function $\tilde{F}_{13}(\rho,\beta)$, which has been expanded in the functions Ψ_{1mn}.

Here, a regular solution of the boundary-value problem in question has been obtained. Any of problems (2.6), (2.7)*a*, (2.8)*a*, (3.15) can be solved for a CCP in exactly the same way. In this case, the convergence of the series representing the solution can be proved. More specifically, it is possible to construct a uniformly converging binary number series with positive terms which, in the domain $\bar{\Omega}$, will dominate the functional series representing the components of the displacement vector and their first and second derivatives. The construction of such a numerical series and the proof of the uniqueness of the solution of problems (2.6), (2.7)*a*, (2.8)*a*, (3.15) is carried out in the same way as previously [7]. As far as the boundary conditions are concerned, when ($\rho = \rho_j$, $\beta = \beta_j$) on just one of the lateral surfaces of the CCP, for example, when $\rho = \rho_0$ and conditions (2.7)*b* are specified when $\rho = \rho_0$, everything can be proved in the same way as in the case of problems (2.6), (2.7)*a*, (2.8)*a*, (3.15) if the system of functions which is generated by problem (3.3), (3.4)*b* when $\rho = \rho_0$, (3.4)*a* when $\rho = \rho_1$, (3.5)*a* is complete. Without touching on the question of investigating the completeness of this system of functions, we merely note that the system of functions generated by the boundary-value problem (3.3), (3.4)*a*, (3.5)*a* is complete [5].

It turns out that for any boundary-value problem from the class of boundary-value problems (2.6)–(2.8), (3.15) to be solved, it is always possible to reduce it to a sum of two such boundary-value problems in each of which the constants appearing in the expression for Φ_{lmn} ($l = 1, 2, 3$) are the same as in relations (4.4) and should be determined from a linear equation and a system of two linear algebraic equations with two unknowns (the above-mentioned "reduction" is more complex for boundary-value problems in which displacements are specified in one of the planes, and arbitrary conditions from the remaining two are specified in the other plane).

We will now illustrate what has been said taking the example of problem (2.6)–(2.8), (2.9)*b* when $\alpha = 0$ and (2.9)*c* when $\alpha = \alpha_1$ (we call this Problem R_0). It can be represented in the form of the sum of two problems: Problem R_1 which differs from Problem R_0 only in that the boundary conditions when $\alpha = \alpha_1$ have the form

$$u = 0, \quad A^{(\alpha)} = 0, \quad w = 0 \tag{4.5}$$

and Problem R_2 which differs from Problem R_0 in the boundary conditions both when $\alpha = 0$ and when $\alpha = \alpha_1$. In particular, when $\alpha = 0$, conditions (4.5) are satisfied, and, when $\alpha = \alpha_1$, the functions u and w are the same as in Problem R_0, and the function f_1, which is the value of $\upsilon(\rho, \alpha_1, \beta)$, taken from the solution of Problem R_1, is subtracted from the expression for υ.

It can be seen that the reduction of an arbitrary boundary-value problem to the sum of two simpler problems leads to the fact that, in each of the two problems, either the symmetry conditions

$$\upsilon = 0, \quad A^{(\beta)} = 0, \quad A^{(\rho)} = 0$$

or the antisymmetry conditions (4.5) are satisfied in one of the boundary planes $\alpha = \alpha_j$. Note that the symmetry and antisymmetry conditions are conditions for the continuation of the solution across the $\alpha = 0$ or $\alpha = \alpha_1$ plane.

In conclusion, we point out that, although the coordinate surfaces of one system of coordinates or another enable us to treat the elastic equilibrium of bodies of different shape, the form of the solution remains unchanged. The shape of an elastic body is solely determined by the form of the parameters h and H.

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